# The Advanced Research WRF (ARW) Dynamics Solver

# ARW Dynamical Solver

– Terrain representation
– Vertical coordinate
– Equations / variables
– Time integration scheme
– Grid staggering
– Advection scheme
– Time step parameters
– Filters
– Boundary conditions
– Nesting
– Map projections

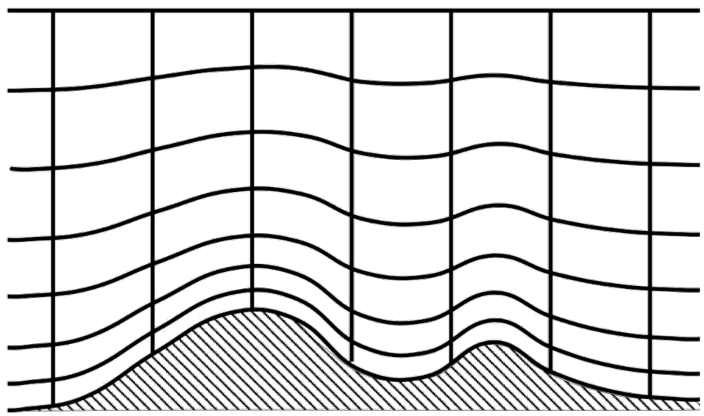
# Vertical Coordinate and Prognostic Variables

Hydrostatic pressure $\pi$

Column mass (per unit area) $\mu = \pi_s - \pi_t$

Vertical coordinate $\eta = \dfrac{(\pi - \pi_t)}{\mu}$

Layer mass (per unit area) $\mu \Delta\eta = \Delta\pi = g\rho\Delta z$



Conserved state (prognostic) variables:

$$\mu, \quad U = \mu u, \quad V = \mu v, \quad W = \mu w, \quad \Theta = \mu\theta$$

Non-conserved state variable: $\phi = gz$

# 2D Flux-Form Moist Equations in ARW

Moist Equations:

$$\frac{\partial U}{\partial t} + \alpha\mu_d \frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} = -\frac{\partial Uu}{\partial x} - \frac{\partial \Omega u}{\partial \eta}$$

$$\frac{\partial W}{\partial t} + g\left(\mu_d - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\right) = -\frac{\partial Uw}{\partial x} - \frac{\partial \Omega w}{\partial \eta}$$

$$\frac{\partial \mu_d}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial \Omega}{\partial \eta} = 0$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial U\theta}{\partial x} + \frac{\partial \Omega\theta}{\partial \eta} = \mu Q$$

$$\frac{d\phi}{dt} = gw$$

$$\frac{\partial(\mu_d q_{v,l})}{\partial t} + \frac{\partial(U q_{v,l})}{\partial x} + \frac{\partial(\Omega q_{v,l})}{\partial \eta} = \mu Q_{v,l}$$

Diagnostic relations:

$$\frac{\partial \phi}{\partial \eta} = -\alpha_d \mu_d, \qquad p = \left(\frac{R\Theta}{p_o \mu_d \alpha_v}\right)^{\gamma}$$

# Time Integration in ARW

## 3rd Order Runge-Kutta time integration

advance $\phi^t \rightarrow \phi^{t+\Delta t}$

$$\phi^* = \phi^t + \frac{\Delta t}{3} R\left(\phi^t\right)$$

$$\phi^{**} = \phi^t + \frac{\Delta t}{2} R\left(\phi^*\right)$$

$$\phi^{t+\Delta t} = \phi^t + \Delta t\, R\left(\phi^{**}\right)$$

Amplification factor $\phi_t = i k \phi; \quad \phi^{n+1} = A \phi^n; \quad |A| = 1 - \frac{(k\Delta t)^4}{24}$

# WRF ARW Model Integration Procedure

Begin time step

- Runge-Kutta loop (steps 1, 2, and 3)
  - (i) advection, p-grad, buoyancy using ($\phi^t$, $\phi^*$, $\phi^{**}$)
  - (ii) physics if step 1, save for steps 2 and 3
  - (iii) mixing, other non-RK dynamics, save…
  - (iv) assemble dynamics tendencies
  - Acoustic step loop
    - (i) advance U,V, then μ, Θ, then w, $\phi$
    - (ii) time-average U,V, Ω
  - End acoustic loop
  - Advance scalars using time-averaged U,V, Ω
- End Runge-Kutta loop
- Adjustment physics (currently microphysics)

End time step

# Flux-Form Perturbation Equations

Introduce the perturbation variables:

$$\phi = \bar{\phi}(\bar{z}) + \phi', \mu = \bar{\mu}(\bar{z}) + \mu';$$
$$p = \bar{p}(\bar{z}) + p', \alpha = \bar{\alpha}(\bar{z}) + \alpha'$$

Note – $\phi = \bar{\phi}(\bar{z}) = \bar{\phi}(x, y, \eta)$,

likewise $\bar{p}(x, y, \eta), \bar{\alpha}(x, y, \eta)$

Reduces horizontal pressure-gradient errors.

For small time steps, recast variables as perturbations from time t

$$U' = U'^{\,t} + U'', \; V' = V'^{\,t} + V'', \; W' = W'^{\,t} + W'',$$
$$\Theta' = \Theta'^{\,t} + \Theta'', \; \mu' = \mu'^{\,t} + \mu'', \; \phi' = \phi'^{\,t} + \phi'';$$
$$p' = p'^{\,t} + p'', \; \alpha' = \alpha'^{\,t} + \alpha''$$

Allows vertical pressure gradient to be expressed in terms of $\phi$''.

# Small Time Step Integration of Acoustic/Gravity Wave Terms

(Without expanding variables into perturbation form)

$$U^{\tau+\Delta\tau} \qquad \frac{\partial U}{\partial t} + \left( \mu_d \alpha \frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta} \frac{\partial \phi}{\partial x} \right)^{\tau} = R_U^t$$

$$\mu_d^{\tau+\Delta\tau} \quad \Omega^{\tau} \qquad \frac{\partial \mu_d}{\partial t} + \frac{\partial U}{\partial x}^{\tau+\Delta\tau} + \frac{\partial \Omega}{\partial \eta}^{\tau+\Delta\tau} = 0$$

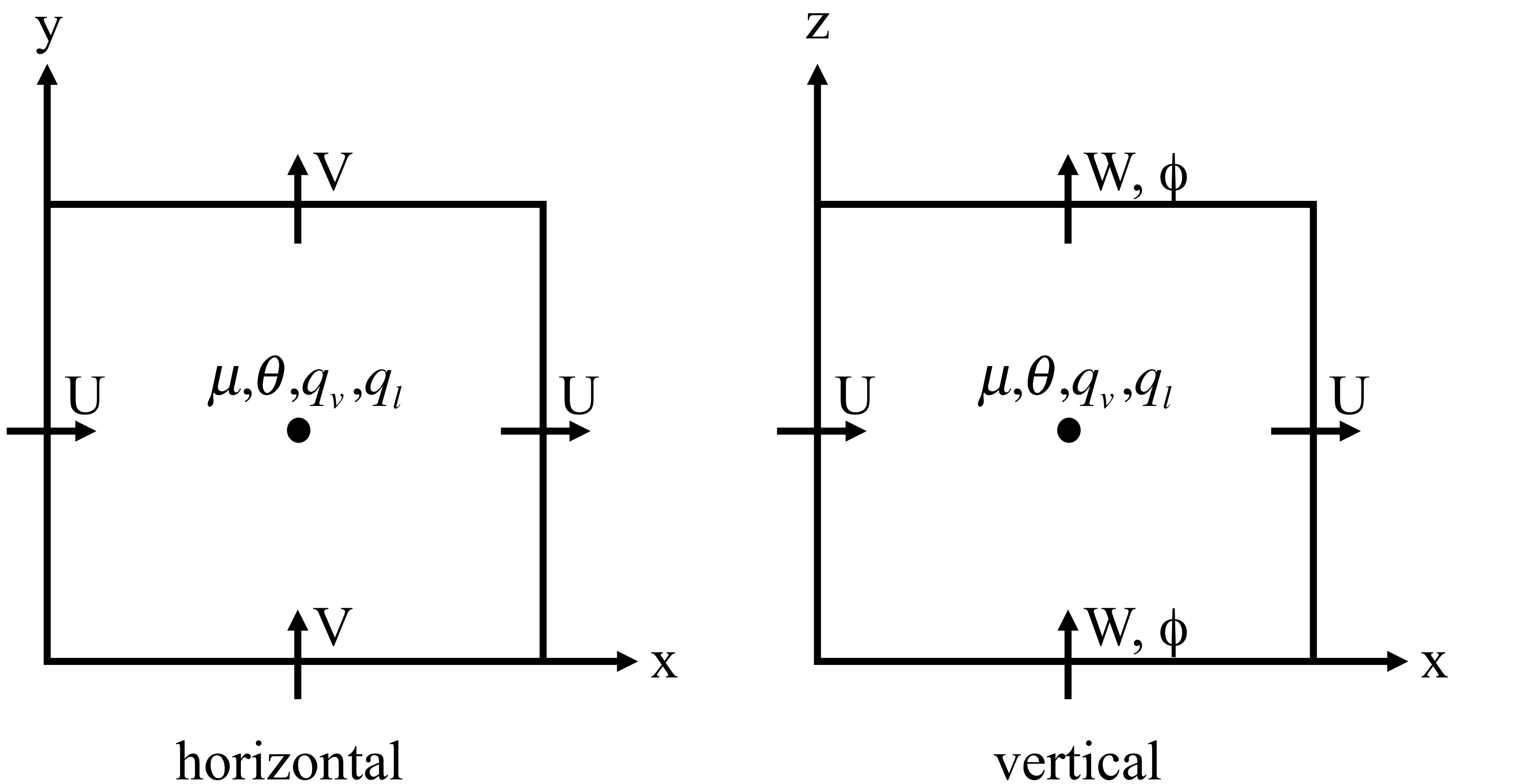
$$\Theta^{\tau+\Delta\tau} \qquad \frac{\partial \Theta}{\partial t} + \left( \frac{\partial U \theta^t}{\partial x} + \frac{\partial \Omega \theta^t}{\partial \eta} \right)^{\tau+\Delta\tau} = R_\Theta^t$$

$$\begin{matrix} W^{\tau+\Delta\tau} \\ \phi^{\tau+\Delta\tau} \end{matrix} \quad \begin{cases} \dfrac{\partial W}{\partial t} + g \overline{\left( \mu_d - \dfrac{\alpha}{\alpha_d} \dfrac{\partial p}{\partial \eta} \right)}^{\tau} = R_W^t \\ \mu_d^t \dfrac{\partial \phi}{\partial t} + U^{\tau+\Delta\tau} \dfrac{\partial \phi}{\partial x}^t + \Omega^{\tau+\Delta\tau} \dfrac{\partial \phi}{\partial \eta}^t - g \overline{W}^{\tau} = R_\phi^t \end{cases}$$

- Forward-backward differencing on $U$, $\Theta$, and $\mu$ equations
- Vertically implicit differencing on $W$ and $\phi$ equations

# ARW model, grid staggering

C-grid staggering



# Advection in the ARW Model

2nd, 3rd, 4th, 5th and 6th order centered and upwind-biased schemes are available in the ARW model.

Example: 5th order scheme

$$\frac{\partial(U\psi)}{\partial x} = \frac{1}{\Delta x}\left(F_{i+\frac{1}{2}}(U\psi) - F_{i-\frac{1}{2}}(U\psi)\right)$$

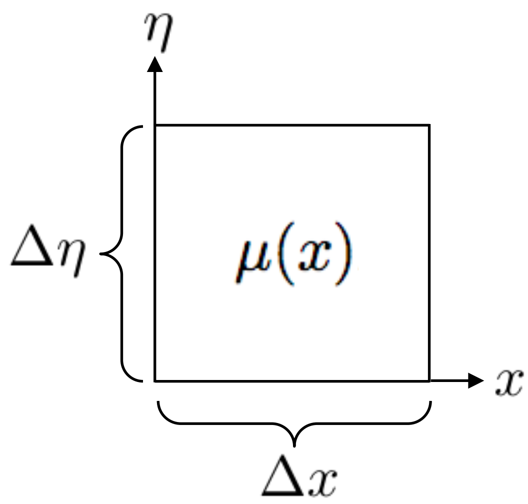
where

$$F_{i-\frac{1}{2}}(U\psi) = U_{i-\frac{1}{2}}\left\{\frac{37}{60}(\psi_i + \psi_{i-1}) - \frac{2}{15}(\psi_{i+1} + \psi_{i-2}) + \frac{1}{60}(\psi_{i+2} + \psi_{i-3})\right\}$$

$$-sign(1,U)\,\frac{1}{60}\left\{(\psi_{i+2} - \psi_{i-3}) - 5(\psi_{i+1} - \psi_{i-2}) + 10(\psi_i - \psi_{i-1})\right\}$$

# Mass Conservation in the ARW Model

control volume
(2D example)

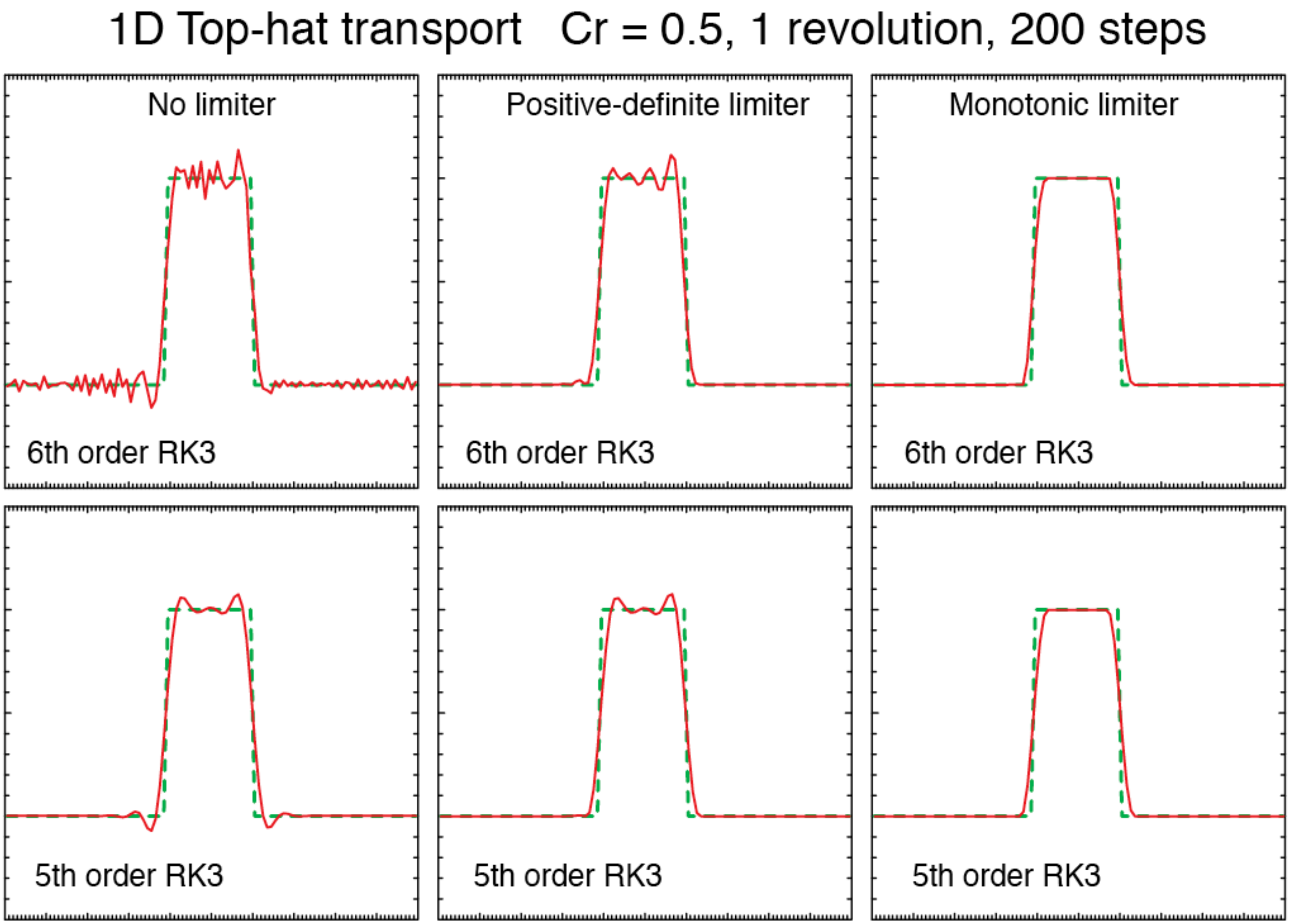



Mass in a control volume is proportional to

$$(\Delta x \Delta \eta)(\mu)^t$$

since $\mu(x)\Delta\eta = \Delta\pi = -g\rho\Delta z$

# PD/Monotonic Limiters in ARW - 1D Example
# Top-Hat Advection



# ARW Model: Dynamics Parameters

## 3rd order Runge-Kutta time step

Courant number limited, 1D: $C_r = \dfrac{U\Delta t}{\Delta x} < 1.43$

Generally stable using a timestep approximately twice as large as used in a leapfrog model.

## Acoustic time step

2D horizontal Courant number limited: $C_r = \dfrac{C_s \Delta\tau}{\Delta h} < \dfrac{1}{\sqrt{2}}$

$$\Delta\tau_{sound} = \Delta t_{RK} / (\text{number of acoustic steps})$$

## Guidelines for time step

*Δt* in seconds should be about 6**Δx* (grid size in kilometers). Larger *Δt* can be used in smaller-scale dry situations, but *time_step_sound* (default = 4) should increase proportionately if larger *Δt* is used.

# ARW Filters: Vertical Velocity Damping

Purpose: damp anomalously-large vertical velocities

(usually associated with anomalous physics tendencies)

Additional term:

$$\partial_t W = \ldots \underline{- \mu_d \, \mathrm{sign}(W) \gamma_w (Cr - Cr_\beta)}$$

$$Cr = \left| \frac{\Omega dt}{\mu d\eta} \right|$$

$Cr_\beta$ = 1.0 typical value (default)

$\gamma_w$ = 0.3 m/s$^2$ recommended (default)

# WRF ARW Model Integration Procedure

Begin time step

Runge-Kutta loop (steps 1, 2, and 3)

(i) advection, p-grad, buoyancy, physics

Acoustic step loop

(i) advance U,V (Fourier Filter U,V)

(ii) advance μ, Θ (Fourier Filter μ, Θ)

(iii) advance w, ϕ (Fourier Filter w, ϕ)

End acoustic loop

Advance scalars (Fourier Filter Sc)

End Runge-Kutta loop

Adjustment physics (currently microphysics) (Fourier Filter Sc)

End time step

Timestep limited by minimum Δx outside of polar-filter region.
Monotonic and PD transport is not available for global model.